ELSEVIER Journal of Geometry and Physics 29 (1999) 243–259

JOURNAL OF GEOMETRY AND PHYSICS

Statistical mechanics of non-stretching elastica in three-dimensional space

Shigeki Matsutani

2-4-11 Sairenji, Niihama, Ehime 792, Japan

Received 20 April 1998

Abstract

Recently by using path integral method and theory of soliton, a new calculation scheme of a partition function of an immersion object has been proposed [J. Phys. A 31 (1998) 2705–2725]. In this paper, the scheme to elastica (space curve with the Bernoulli–Euler functional) immersed in three-dimensional space $\mathbb{R}^3$ as a physical model in polymer science is applied. It is shown that the nonlinear Schrödinger and complex modified Korteweg–de Vries hierarchies naturally appear to express the functional space of the partition function. In other words, it is shown that the configuration space of an elastica immersed in $\mathbb{R}^3$ can be classified by these equations. Then the partition function is reduced to an ordinary integral over the orbit space of these hierarchies.

Subj. Class.: Differential geometry; Classical mechanics
1991 MSC: 53A04; 82B05; 82D60; 35Q53; 58E20
Keywords: Space curve; Statistical mechanics; Soliton theory

1. Introduction

Elastica problem in two-dimensional space $\mathbb{R}^2$ has long history [1,2]. It is known that by observing a shape of thin elastic beam, James Bernoulli named the shape elastica [1,2]. It might be regarded as birth of the elastica problem and germination of the mathematical physics, including differential geometry, harmonic map theory, elliptic function theory, mode analysis, nonlinear science, elliptic differential theory, algebraic analysis and so on, because by studying the elastica problem, such theories were generated and developed [1,2]. The elastica in $\mathbb{R}^2$ [1,2] is defined as a curve with the Bernoulli–Euler functional

$$E = \int \mathrm{d}s k^2, \tag{1.1}$$

PII S0393-0440(98)00042-4

where k is its curvature. Euler applied the minimal principle to this problem and investigated its shape or the functional space $k(s)$. Then shapes of the elastica, functional space of classical static elastica, were completely classified in the eighteenth century, which is an exactly solved case in the harmonic map theory [1,2]. It is known that its moduli, points of the solution space, are expressed by the moduli of the elliptic function [1,2].

Recently, I presented a new calculation scheme of a partition function of nonstretching elasticas in $\mathbb{R}^2$. The partition function is formally defined as

$$Z = \int DX \mathrm{e}^{-\beta \int \mathrm{d}sk^2}, \tag{1.2}$$

where DX is the Feynman measure for an affine vector of a point of the elastica X and β is the inverse of temperature. Goldstein and Petrich discovered that the virtual motion of nonstretching curve obeys the modified Korteweg–de Vries (MKdV) equation

$$\partial_t k + \tfrac{3}{2} k^2 \partial_s k + \partial_s^3 k = 0, \tag{1.3}$$

and its hierarchy [4,5]. Using the Goldstein–Petrich scheme, I found that the functional space of the partition function (1.2) are completely represented by the MKdV equation (1.3) and there naturally appears the MKdV hierarchy. The MKdV flows conserve the energy functional (1.1). Thus the functional space (1.2) is classified by the solutions of the MKdV equation (1.3) and its hierarchy. The points of the functional space of the statistical mechanics of the elastica are expressed by the hyperelliptic functions and their moduli because a solution of the MKdV equation is expressed by the hyperelliptic function and is characterized by modulus of the corresponding hyperelliptic curve [6–9]. Then the integral in the partition function (1.2) is reduced to the ordinary (Riemannian) integral over the solution space of the MKdV equation, or over the Jacobi varieties of the hyperelliptic curves [6–9].

Here I will comment upon the result of [3] from the viewpoint of soliton theory. Even though there were so many proposals on the relations between soliton theory and physical problems, few cases are recognized which are models of observable problems of physics and whose relations are not just an approximation. In fact, though virtual motion of nonstretching curve are investigated by many authors [4,5,10–15, references therein], it has not been clarified that they are concerned with the physical problem, such as an elastica, except reinterpretation of curve problem as spin problem [10]. For example, the motion of elastica, which is derived from an action constituted by (1.1) and kinetic term, is not expressed by the soliton equation in general [11–13]. The kinetic term disturbs the integrability though the motion approximately obeys soliton equations [11–13].

Thus it is very surprising that (1.2) with (1.1) are completely related to soliton theory in Ref. [3]. By investigating the statistical mechanics of elastica, I reproduced the algebraic relations in the generators in infinite dimensional linear system as differential ring [8,9], which corresponds to a subspace of the universal Grassmannian manifold (UGM) [8,9], and virtual dynamics in the Jacobi varieties of the hyperelliptic curves [6,7], as Euler found the elliptic function theory and its moduli by studying mechanics of elastica.

After the studies on elastica in $\mathbb{R}^2$ [3], I applied this method to the Willmore surface in three-dimensional space $\mathbb{R}^3$ [16]. Instead of the MKdV equation, there appears the modified Novikov–Veselov equation which classifies the functional space of the partition function of the Willmore surface.

In this article, I will investigate a partition function of an elastica in $\mathbb{R}^3$ with the energy functional

$$E = \int \mathrm{d}s |k|^2, \tag{1.4}$$

where k is a complex curvature of the elastica in $\mathbb{R}^3$. This energy functional (1.4) is investigated from the point of view of the harmonic map theory [15 and references therein], which corresponds to the system of the zero temperature of the statistical mechanics of the elastica. I will require that the elastica does not stretch even for finite temperature as in [3].

Then it will be shown that the partition function of elastica in $\mathbb{R}^3$ with the energy (1.4) can be also evaluated. Due to the nonstretching condition, instead of Goldstein–Petrich scheme of the elastica in $\mathbb{R}^2$ [3–5], I will use the Langer–Perline scheme for a curve in $\mathbb{R}^3$ [14,15]. By using them, I will show that the nonlinear Schrödinger (NLS) hierarchy and the complex MKdV (CMKdV) hierarchy naturally appears in the calculation of the partition function. Thus as first purpose of this article, I will reveal the physical origin of the NLS and the CMKdV hierarchies.

On the other hand, it should be noted that (1.4) sometimes appear in a polymer physics as an action of large polymer [17,18]. Such a model is referred as elastic chain model. Thus investigation of the partition function of (1.4) is very natural from physical demand. Due to such a physical background, in Ref. [3], I investigated the statistical mechanics of the elastica (1.1), but elastica of (1.1) is in a plane while a physical polymer is in three-dimensional space. Hence as a more physical problem, I will study the partition function of (1.4) here. The polymer physics is a very complex problem. Due to the complexity, investigation of properties of polymers is not simple in general. However, it sometimes can be exactly performed owing to deep symmetry [17–19]. In fact, an exact partition function of elastic chain with the energy functional (1.4) was already obtained by Saitô et al. [19] using the path integral. However, they paid no attention upon isometry condition as thermal fluctuation of the path integration even though they required isometry condition after all computations; they summed allover configuration space without isometry condition rather than over restricted functional space. It should be noted that the constraint does not commute with such evaluation of the partition function in general. Thus it implies that their partition function is of a stretchy polymer with energy functional (1.4).

Thus as another limit, it is of interest to investigate the partition function with the energy (1.4) under the isometry condition. By considering such theoretical situations, second purpose of this article, which is identified with the first one, is to investigate the partition function of a nonstretching space curve with the energy functional (1.4). Thus I believe that this study influences the polymer science.

Furthermore, a space curve in $\mathbb{R}^3$ also interests us from the viewpoint of the string theory [3,11,16,18,20]. According to Kholodenko's review of a large polymer [18], statistical mechanics of a polymer model is closely connected with the mathematical science, string theory and quantum field theory. Grinevich and Schmidt investigated closed condition of a space curve obeying the NLS equation because a kind of its complexification becomes a surface with Kähler metric [21]. Since a surface with Kähler metric can be regarded as world-sheet of a string, this problem is also associated with the string theory [22]. (However, as I will mention later, it should be noted that the elastica absolutely differs from a string in the string theory, even though it influences the theory [3,11,16,18,20].) Thus although it is not a main purpose, third hidden purpose of this article is to investigate the moduli of nonstretching curve in $\mathbb{R}^3$ by taking into the consideration of such relation as a generalization to the surface problem [16,20,21].

The organization of this article is as follows. In Section 2, I will evaluate the partition function of nonstretching elastica in $\mathbb{R}^3$. Section 3 gives a discussion of the results.

2. Partition function of nonstretching elastica in $\mathbb{R}^3$

I will denote by C a shape of a closed elastica (a real one-dimensional closed curve) immersed in three-dimensional space $\mathbb{R}^3$ and by $\mathbf{X}(s) = (X^1, X^2, X^3)$ its affine vector [23]

$$\begin{aligned} &S^1 \ni s \mapsto X(s) \in C \subset \mathbb{R}^3, \\ &\partial_s^n \mathbf{X}(s+L) = \partial_s^n \mathbf{X}(s) \quad (n \in \mathbb{N} + \{0\}), \end{aligned} \tag{2.1}$$

where L is the length of the elastica, s a parameter of the curve and $\mathbb{N}$ is natural number. As I will mention its physical model, it can be regarded as a closed polymer in $\mathbb{R}^3$; its center axis is a space curve C. Here I will fix the metric of the curve C induced from the natural metric of $\mathbb{R}^3$;

$$\mathrm{d}s = \sqrt{\mathrm{d}\mathbf{X}\,\mathrm{d}\mathbf{X}}. \tag{2.2}$$

There is the orthonormal system along C, $(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2)$ with fixing $\mathbf{n}_0$ as the tangent unit vector; $\mathbf{n}_0 = \partial_s \mathbf{X}$, where $\partial_s := \partial/\partial s$. I make them, first, satisfy the Frenet–Serret relation [23,24]

$$\partial_s \begin{pmatrix} \mathbf{n}_0 \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n}_0 \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{pmatrix}, \tag{2.3}$$

Here k is the curvature, τ is the Frenet–Serret torsion and they are functions of only s. I rotate the orthonormal frame SO(2) fixing $\mathbf{a}_0 := \mathbf{n}_0$ so that $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ is obtained as [23–27],

$$\partial_s \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}, \tag{2.4}$$

where $\kappa_1 := k\cos\theta$, $\kappa_2 := k\sin\theta$ and

$$\theta(s) := \int_{s_0}^{s} \tau(s')\,\mathrm{d}s'. \tag{2.5}$$

For convenience, I will introduce a complex curvature, $\kappa := \kappa_1 + \mathrm{i}\kappa_2 = k\mathrm{e}^{\mathrm{i}\theta}$.

In this paper, I will deal with a nonstretching elastica in $\mathbb{R}^3$ with the energy functional

$$E = \int_0^L \mathrm{d}s\, |\kappa|^2, \tag{2.6}$$

which I will also call Bernoulli–Euler functional [1–3]. For a thin elastic rod, a potential term like (2.6) appears in its free energy due to its thickness and an elastica of (2.6) is an idealized model of such thin elastic rod [7]. However, it is worthwhile noting that in general, there appear other potential terms in the energy functional for a general elastic rod. For example, there might appear elastic torsion term (for twist), stretching term and so on. An elastica is usually defined as a curve realized as a stationary point of an energy functional related to an elastic rod, at least, in the meaning of the classical mechanics. Hence, the word "elastica" sometimes has ambiguity. Depending upon the potential term, its shape might belong to individual class. Thus reader should not confuse the word "elastica" with another one in another context. In this paper, the word "elastica" is used for only a curve with the Bernoulli–Euler functional (2.6).

As I stated in Section 1, the elastica I deal with here can be regarded as a model of a polymer which can freely rotate around its center axis but does not stretch and is forced by the potential (2.6) [17,18]. In other words, I assume that the force from the elastic torsion (twist) can be negligible but stretching cannot. Furthermore, I will neglect the kinetic term of the elastica. Physically speaking, I will consider the polymers in the liquid whose temperature is determined and viscosity is very large. I also suppose that each polymer behaves independently and interaction among them are neglected.

As I stated in Ref. [3], a reader should not confuse an elastica with a "string" in a string theory [22]; they are absolutely different. The word "string" in a string theory is a technical term in the particle physics and is not string in ordinary meaning. For example, string (wire) in the guitar resembles elastica rather than "string" in the string theory [2]. In fact the potential term of the action of "string" in the string theory is harmonic $(\partial_s \mathbf{X})^2$ while (2.6) is expressed by a biharmonic type $(\partial_s^2 \mathbf{X})^2$, which comes from the effects of the thickness of the wire.

Now I will start to consider the statistical mechanics of elastica (2.6). Let the elastica be closed and preserve its local infinitesimal length for even thermal fluctuation. It does not stretch. Under the conditions, I will consider a partition function of the elastica given as [3,18]

$$\mathcal{Z} = \int D\mathbf{X} \exp\left(-\beta \int_0^L \mathrm{d}s |\kappa|^2 \right), \tag{2.7}$$

where β is the inverse of temperature. Following the calculation scheme which I proposed in Refs. [3,16], I will evaluate the partition function (2.7) under the nonstretching condition.

However, there is trivial affine symmetry of the centroid and direction of the elastica and the partition function naturally diverges [3]. For an affine transformation (translation and rotation $g \in \mathrm{SO}(3)$), $\mathbf{X}(s) \to \mathbf{X}_0 + g\mathbf{X}(s)$ ($\mathbf{X}_0$ and g are constants of s), the curvature κ and the Bernoulli–Euler functional (2.6) does nto change; this is a gauge freedom and the energy functional (2.6) has infinitely degenerate states. In the path integral method, I must sum over all possible states, $\mathcal{Z}$ includes the integration over $\mathbb{R}^3$ and naturally diverges. As well as the arguments in Refs. [3,16], I will regularize it

$$\mathcal{Z}_{\mathrm{reg}} = \frac{\mathcal{Z}}{\mathrm{Vol(Aff)}}, \tag{2.8}$$

where Vol(Aff) is the volume of the space related to the affine transformation. By this regularization, I can concentrate on the classification of shapes of elastica. As I have divided the configuration space $\mathbf{X}(s)$ by the equivalent class, I should write a representative element of the quotient space, $[\mathbf{X}(s)]$. However for brevity, I will go on to write it $\mathbf{X}(s)$ but I assume that from this point $\mathbf{X}(s)$ is an element of the quotient space.

Next I will investigate the condition preserving local length even for the thermal fluctuation. I will expand the affine vector around the point which is an extremum point of the Bernoulli–Euler functional (2.6). I will call the point quasi-classical point according to the quasi-classical method in path integral [3,16,28]. In the path integral, I must pay attention to the higher perturbations of ϵ in order to obtain an exact result. Hence I will assume that $\mathbf{X}$ is parametrized by a deformation parameter t. I will express a perturbed affine vector $\mathbf{X}$ around an extremum point $\mathbf{X}_{\mathrm{qcl}}$ in the partition function (2.8) as [3,14–16]

$$\mathbf{X}(s,t) := \mathrm{e}^{\epsilon\partial_t}\mathbf{X}\mathrm{qcl}(s,t), \qquad \epsilon\partial_t\mathbf{X}_{\mathrm{qcl}} = \mathbf{X}_{\mathrm{qcl}} - \mathbf{X} + \mathrm{O}(\epsilon^2) \tag{2.9}$$

with the relation

$$\begin{aligned} &\partial_t\mathbf{X}_{\mathrm{qcl}} = u_0\mathbf{a}_0 + u_1\mathbf{a}_1 + u_2\mathbf{a}_2,\\ &u_a(L) = u_a(0) \quad (a = 0,1,2), \end{aligned} \tag{2.10}$$

where u's are real functions of s and t. I can regard (2.10) as virtual dynamics of the curve describing the thermal fluctuation [3] and thus I will call the deformation parameter t "virtual times" by distinguishing real motion of elastica in dynamical problem [11–13] or nonequilibrium statistical mechanics; in this paper, I am dealing with only object in the framework of the equilibrium statistical mechanics and thus readers should not confuse the variable t with physical times.

As in Refs. [3,16], even in thermal fluctuation, I will assume that elastica does not stretch; I will restrict the functional space $\mathbf{X}(s)$. Due to the isometry condition, I require $[\partial_t, \partial_s] = 0$ for $\mathbf{X}$ [3–5,14–16]. Since $\mathrm{d}s_{\mathrm{qcl}} := \sqrt{\partial_x\mathbf{X}_{\mathrm{qcl}}\partial_s\mathbf{X}_{\mathrm{qcl}}}\,\mathrm{d}s$, the isometry condition exactly preserves, $\mathrm{d}s \equiv \mathrm{d}s_{\mathrm{qcl}}$ if $[\partial_t, \partial_s] = 0$.

Let us compute the nonstretching condition $[\partial_t, \partial_s]\mathbf{X}_{\mathrm{qcl}} = 0$. By introducing "velocities" $(\partial_t\phi_1, \partial_t\phi_2)$, it is evaluated as

$$\partial_t \mathbf{a}_0 \equiv \partial_t \partial_s \mathbf{X}_{\mathrm{qcl}} := (\partial_t \phi_2)\mathbf{a}_1 - (\partial_t \phi_1)\mathbf{a}_2, \tag{2.11}$$

$$\partial_s \partial_t \mathbf{X}_{\mathrm{qcl}} = (\partial_s u_0 - \kappa_1 u_1 - \kappa_2 u_2)\mathbf{a}_0 + (\partial_s u_1 + \kappa_1 u_0)\mathbf{a}_1 + (\partial_s u_2 + \kappa_2 u_0)\mathbf{a}_2. \tag{2.12}$$

From the condition, I have the relation between $\partial_t \phi_{\mathrm{c}} (\phi_{\mathrm{c}} := \phi_1 + \mathrm{i}\phi_2)$ and a complex "velocity", $u_{\mathrm{c}} := u_1 + \mathrm{i}u_2$,

$$\begin{aligned}\partial_t \phi_{\mathrm{c}} &= \mathrm{i}(\kappa_{\mathrm{qcl}} u_0 + \partial_s u_{\mathrm{c}}) \\ &= \mathrm{i}(\kappa_{\mathrm{qcl}} \partial_s^{-1} \mathrm{Re}((\overline{\kappa_{\mathrm{qcl}}} u_{\mathrm{c}}) + \partial_s u_{\mathrm{c}})) =: Q(u_{\mathrm{c}}).\end{aligned} \tag{2.13}$$

Here I have used the notation $\kappa_{\mathrm{qcl}} := \kappa_1 + \mathrm{i}\kappa_2$ and I introduced the pseudo-differential operator ∂_s^{-1},

$$\begin{aligned}\partial_s u_0 &= \mathrm{Re}(\overline{\kappa_{\mathrm{qcl}}} u_{\mathrm{c}}) = (\kappa_{\mathrm{qcl}} \overline{u_{\mathrm{c}}} + \overline{\kappa_{\mathrm{qcl}}} u_{\mathrm{c}})/2, \\ u_0 &= \partial_s^{-1} \mathrm{Re}(\overline{\kappa_{\mathrm{qcl}}} u_{\mathrm{c}}) = \int^s \mathrm{d}s' \mathrm{Re}(\overline{\kappa_{\mathrm{qcl}}(s')} u_{\mathrm{c}}(s')).\end{aligned} \tag{2.14}$$

In order to find the connection between ϕ_{c} and κ, I will also investigate the fluctuation of $\mathbf{a}_a (a = 1, 2)$. Noting $\mathbf{a}_0$, differentiation of $\mathbf{a}_a (a = 1, 2)$ by t must have the form

$$\partial_t \mathbf{a}_1 = -\partial_t \phi_2 \mathbf{a}_0 - v \mathbf{a}_2, \qquad \partial_t \mathbf{a}_2 = \partial_t \phi_1 \mathbf{a}_0 + v \mathbf{a}_2, \tag{2.15}$$

where v means the rotation in the plane spanned by $\mathbf{a}_a (a = 1, 2)$. By requirement of the isometry, the virtual dynamics of $\mathbf{a}_a$ is constrained as $[\partial_t, \partial_s]\mathbf{a}_a = 0$ $(a = 1, 2)$,

$$\begin{aligned}-\partial_s \partial_t \mathbf{a}_1 &= (\partial_s \partial_t \phi_2 - \kappa_2 v)\mathbf{a}_0 + (\partial_t \phi_2 \kappa_1)\mathbf{a}_1 + (\partial_t \phi_2 \kappa_1 + \partial_s v)\mathbf{a}_2, \\ -\partial_s \partial_t \mathbf{a}_2 &= -(\partial_s \partial_t \phi_1 - \kappa_2 v)\mathbf{a}_0 - (\partial_t \phi_1 \kappa_1)\mathbf{a}_1 - (\partial_t \phi_1 \kappa_2 + \partial_s v)\mathbf{a}_2, \\ -\partial_t \partial_s \mathbf{a}_1 &= \partial_t \kappa_1 \mathbf{a}_0 + (\kappa_1 \partial_t \phi_2)\mathbf{a}_1 - (\kappa_1 \partial_t \phi_2)\mathbf{a}_2, \\ -\partial_t \partial_s \mathbf{a}_2 &= \partial_t \kappa_2 \mathbf{a}_0 + (\kappa_2 \partial_t \phi_2)\mathbf{a}_1 - (\kappa_2 \partial_t \phi_2)\mathbf{a}_2.\end{aligned} \tag{2.16}$$

Hence I have the relation [14,15],

$$\partial_t \kappa_{\mathrm{qcl}} = -Q(\partial_t \phi). \tag{2.17}$$

Accordingly, I have the relation between $\partial_t k$ and complex velocity u_{c} as the "equation of motion" of the deformation satisfied with the isometry condition [14,15]

$$\partial_t k_{\mathrm{qcl}} = -Q^2(u_{\mathrm{c}}). \tag{2.18}$$

I will remark that Q^2 is known as the recursion operator of the NLS and the CMKdV equations [14,15].

For this nonstretching deformation, the Bernoulli–Euler functional (2.6) changes as

$$\begin{aligned}\int |\kappa|^2 \,\mathrm{d}s = \int &(|\kappa_{\mathrm{qcl}}|^2 + \epsilon(\overline{\kappa_{\mathrm{qcl}}} \partial_t \kappa_{\mathrm{qcl}} + \kappa_{\mathrm{qcl}} \partial_t \overline{\kappa_{\mathrm{qcl}}})) \\ &+ \epsilon^2((|\partial_t \kappa_{\mathrm{qcl}}|^2 + \overline{\kappa_{\mathrm{qcl}}} \partial_t^2 \kappa_{\mathrm{qcl}} + \kappa_{\mathrm{qcl}} \partial_t^2 \overline{\kappa_{\mathrm{qcl}}}) + \cdots) \,\mathrm{d}s\end{aligned}$$

$$= \int (\kappa_{\rm qcl}^2 - \epsilon(\overline{\kappa_{\rm qcl}} Q^2(u_c) + \kappa_{\rm qcl}\overline{Q^2(u_c)}))\,{\rm d}s + {\rm O}(\epsilon^2)$$

$$=: E_{\rm qcl} + \delta^{(1)} E_{\rm qcl} + \delta^{(2)} E_{\rm qcl} + \cdots \tag{2.19}$$

Since I wish to expand the complex curvature k around the extremum point in the functional space, I will require the extremum condition [3]

$$\delta^{(1)} E_{\rm qcl} = 0. \tag{2.20}$$

In this method, I will sum the weight function over all extremum points. Since they are extremum rather than stationary points, they need not be realized in zero temperature, $\beta \to \infty$.

Noting the relation $\partial_s u_0 = (\overline{\kappa_{\rm qcl}} u_{\rm c} + \kappa_{\kappa{\rm cl}}\overline{u_c})/2$ and above notices, supposed that $\overline{\kappa_{\rm qcl}} Q^2 (u_{\rm c}) + \kappa_{\rm qcl}\overline{Q^2(u_{\rm c})}$ could be regarded as another function $\overline{\kappa_{\rm qcl}} u'_{\rm c} + \kappa_{\rm qcl}\overline{u'_{\rm c}}$ of the variation of the normal direction in (2.14), I might find the relation

$$\int {\rm d}s {\rm Re}(\overline{\kappa_{\rm qcl}} Q^2(u_{\rm c})) \sim \int {\rm d}s {\rm Re}(\overline{\kappa_{\rm qcl}} u'_c) = \int {\rm d}s (\partial_s u'_0) = 0. \tag{2.21}$$

I supposed that the deformation was described by one parameter t. However, there is no requirement that I must go along with only one parameter t to characterize this system. In the calculation of the partition function, one must sum up the weight function over events unless probability occurrence of the events vanishes. I will search for all possible extremum points.

Furthermore, in a microcanonical system at energy E_0, the entropy S of the system is defined as $S := \log Z|_{E=E_0}$ and can be regarded as the logarithm of the volume of the functional space. From primitive consideration, the dimension of the functional space in the statistical physics is related to the degrees of freedom corresponding to E_0 and the degrees of freedom of the elastica are not finite. Thus the dimension of the deformation parameter $\{t\}$ need not be one.

Along the line of the arguments of Ref. [3], I will give up to express the thermal fluctuation using only one parameter t and I will introduce the sequence for "virtual times" $\mathbf{t} := (t_1, t_3, t_5, \ldots, t_{2n+1}, \ldots)$ in this system so that (2.21) is satisfied. I will redefine the fluctuation (2.9) and introduce infinite parameters family, which can sometimes become finite set as I will show later,

$$\mathbf{X}_{\delta \mathbf{t}} = \exp\left((1/\sqrt{\beta}) \sum_{n=0} \delta t_{2n+1} \partial_{t_{2n+1}}\right) \mathbf{X}_{\rm qcl}$$

$$= \mathbf{X}_{\rm qcl} + (1/\sqrt{\beta}) \sum_{n=0} \delta t_{2n+1} \partial_{t_{2n+1}} X_{\rm qcl} + {\rm O}(1/\beta), \tag{2.22}$$

where ϵ was replaced with $(1/\sqrt{\beta})\delta t_{2n+1}$ and $\partial_{t_{2n+1}} X_{\rm qcl}$ is expressed as

$$\partial_{t_{2n+1}} \mathbf{X}_{\rm qcl} = u_0^{(n)} \mathbf{a}_0 + u_1^{(n)} \mathbf{a}_1 + u_2^{(n)} \mathbf{a}_2,$$

$$u_0^{(n)} = \partial_s^{-1} {\rm Re}(\bar{\kappa}_{\rm qcl} u_{\rm c}^{(n)}), \qquad u_{\rm c}^{(n)} = Q^{2n}(u_{\rm c}^{(0)}). \tag{2.23}$$

The virtual equations of motion for the deformation are expressed as

$$\partial_{t_{2n+1}} \kappa = Q^{2n}(u_{\rm c}^{(0)}). \tag{2.24}$$

Thus (2.24) represents the thermal fluctuation which conserves the local length. As in Ref. [3], I have implicitly assumed that $[\partial_{t_{2n+1}}, \partial_{t_{2m+1}}] = 0$. It implies that t_{2n+1} and t_{2m+1} are orthogonal or parallel. In fact, if there is a number n_0 such as $Q^2(u_{\rm c}^{(n_0)}) = \Lambda u_{\rm c}^{(n_0)}$, with a constant value $\Lambda \in \mathbb{R}$, it is concluded that $t_{2m+1} \propto t_{2n_0+1}, m > n_0$. For hierarchy belonging to the KP hierarchy such as the MKdV equation [3,6–9], similar relation holds while $t_{2m+1} (m < n_0)$ are perpendicular with t_{2n_0+1} in the related Jacobi variety [6–9]. As the independent parts are enough to represent the thermal fluctuation, the parallel parts are not necessary and the formal expansion with infinite dimensional parameters in (2.22) is reduced to finite dimensional one.

As I obtained the family of the local-length-preserving curves following the arguments [14,15], I will restrict it by imposing physical requirement. First I will note that there are two manifest symmetries in this system. One exhibits the symmetry of choice of the origin s and another is for the symmetry of U(1) phase of κ; the latter one is the same as the choice of s_0 at the integration (2.5). For the transformation $\kappa(s) \to {\rm e}^{{\rm i}t} \kappa(s - \bar{t})$, the partition function is invariant. Hence the family of the deformations (2.22) must contain these symmetries.

Second it should be noted that the partition function should contain the classical shape at zero temperature $\beta \to \infty$. Here the classical shape I call means a shape which is obtained from the Bernoulli–Euler functional following the minimal principle without thermal fluctuation or $\beta \to \infty$ (see Appendix A). (I will use the term "classical" in the meaning of the analogy between the statistical physics and quantum physics, even though I am dealing only with the ordinary statistical mechanics rather than quantum statistical mechanics.)

Accordingly I will require that the deformation must include trivial symmetries of the system and classical shapes as in Ref. [3,16].

Hence I impose, as the first requirement, that the virtual motions include such manifest symmetries

$$\partial_{\bar{t}_1} \kappa_{\rm qcl} = \partial_s \kappa_{\rm qcl} \quad \text{and} \quad \partial_{t_1} \kappa_{\rm qcl} = {\rm i}\kappa_{\rm qcl}. \tag{2.25}$$

It should be also noted that there is no other manifest symmetry except (2.25). In other words, (2.25) is unique from the point of view of choice of coordinate. Here I will note that t_1 and $\bar{t}_1$ also have the relation $[\partial_{t_1}, \partial_{\bar{t}_1}] = 0$.

Instead of the single sequence of deformation parameters as in Refs. [3,16], I will assign the infinite dimensional parameters in (2.22) to those which fulfill the requirement; $\mathbf{t} := (\{t\}, \{\bar{t}\}) = (t_1, t_3, \ldots, \bar{t}_1, \bar{t}_3, \ldots)$.

$$\partial_{\bar{t}_{2n+1}} \kappa_{\rm qcl} = (-Q^2)^n (\partial_s \kappa_{\rm qcl}), \quad \partial_{\bar{t}_{2n+1}} \kappa_{\rm qcl} = -Q^2 (\partial_{\bar{t}_{2n-1}} \kappa_{\rm qcl}) \quad (n = 1, 2, \ldots). \tag{2.26}$$

$$\partial_{t_{2n+1}} \kappa_{\rm qcl} = (-Q^2)^n ({\rm i}\kappa_{\rm qcl}), \quad \partial_{t_{2n+1}} \kappa_{\rm qcl} = -Q^2 (\partial_{t_{2n-1}} \kappa_{\rm qcl}) \quad (n = 1, 2, \ldots). \tag{2.27}$$

They are the CMKdV and the NLS hierarchies, respectively.

Though the properties of the CMKdV equation might be not well-known, for intrinsically real initial condition $\kappa(s) \in e^{i\alpha_0}\mathbb{R}$ (constant α_0 for s and $\mathbf{t}$), the CMKdV equation is reduced to the MKdV equation. In [14], the CMKdV equation appears in the context of the geometrical realization of the inverse scattering method. Further, it is known that the CMKdV equation has nontrivial N-soliton solutions $N > 1$ [29] and the properties of the CMKdV hierarchy and the CMKdV equation are very regular as I show as follows. Thus the CMKdV equation might be regarded as completely integrable.

In fact, as the NLS hierarchy, a solution of the nth CMKdV equation

$$\partial_{\bar{t}_{2n+1}}\kappa_{\rm qcl} - (-Q^2)^n(\partial_s\kappa_{\rm qcl}) = 0 \tag{2.28}$$

is satisfied with the simultaneous equations by introducing unknown parameter $\bar{t}_{2n-1}$,

$$\begin{aligned} \partial_{\bar{t}_{2n+1}}\kappa_{\rm qcl} - Q^2(\partial_{\bar{t}_{2n-1}}\kappa_{\rm qcl}) &= 0, \\ \partial_{\bar{t}_{2n-1}}\kappa_{\rm qcl} &= (-Q^2)^{n-1}(\partial_s\kappa_{\rm qcl}). \end{aligned} \tag{2.29}$$

By ladder-type calculations with respect to n and $n-1$ of recurrent relation (2.29), it can be proved that the solution space of the higher order equations belonging to the CMKdV hierarchy is also determined by the CMKdV equation

$$\partial_{\bar{t}}\kappa + \tfrac{3}{2}|\kappa|^2\partial_s\kappa + \partial_s^3\kappa = 0. \tag{2.30}$$

In other words, the nontrivial deformation is obtained by the CMKdV equation as the ordinary soliton hierarchy does. (One might have a question why the ladder-type calculation terminates at $\bar{t} = \bar{t}_3$ rather than $\bar{t} = \bar{t}_1$. From (2.25) $\bar{t}_1$ is determined as $\bar{t}_1 \equiv s + s_0$ and thus $\bar{t}_1$ is not an unknown parameter in the sense of (2.29). Eq. (2.30) is a minimal nontrivial equation.)

Similarly, I have the NLS equation to get nontrivial deformation of the NLS hierarchy

$$i\partial_t\kappa + \tfrac{1}{2}|\kappa|^2\kappa + \partial_s^2\kappa = 0. \tag{2.31}$$

For the NLS equation, this reduction can be naturally justified in the Jacobi variety of the hyperelliptic curve as a solution space [6].

Furthermore due to $[\partial_{t_1}, \partial_{\bar{t}_1}] = 0$, (2.29) and similar relation for the NLS equation, the relations $[\partial_{t_{2n+1}}, \partial_{\bar{t}_{2m+1}}] = 0 (n, m \in \mathbb{N})$ and $[\partial_t, \partial_{\bar{t}}] = 0$ are expected. It implies that the NLS equations and the MKdV equations are independent equations. In fact, if one gives the real value κ, κ goes on real in the "time" $\bar{t}$ development of the CMKdV equation (2.30) whereas for the NLS equation (2.31) its "time" t development includes the complex value due to the pure imaginary in the first term in (2.31). The "virtual time" developments of both equations for an initial state differ from each other. Eqs. (2.26) and (2.27) are expected as independent families of deformations which are satisfied with the isometry condition and physical requirements of the trivial symmetries.

As I realized the first physical requirement for the trivial symmetries, I will mention the classical shapes. In (2.31), replacement of the time t with s/C_0 for constant C_0 leads us to obtain Eq. (A.3) for the classical shapes (see Appendix A). Hence the solutions of the NLS hierarchy (2.27) and Eq. (2.31) contain those of the classical shapes. Consequently these deformations (2.26), (2.27), (2.30) and (2.31) are satisfied with the physical requirements I mentioned above and isometry condition.

Here I will comment upon the result of Mohammad and Can [30]. They investigated the "complex MKdV" equation and concluded that it is not a soliton equation. However, their "complex MKdV equation" is expressed as

$$\partial_{\bar{t}}\kappa + \tfrac{1}{2}\partial_s(|\kappa|^2\kappa) + \partial_s^3\kappa = 0, \tag{2.32}$$

which is a kind of "complexification" of the MKdV equation but differs from (2.30). Their result does not directly affect the studies on the integrability of our CMKdV equation (2.30).

As I obtained the isometry deformation which includes the unique manifest symmetries (2.25) and classical shapes, I will consider, here, the partition function (2.7).

Before seeing the measure of the partition function, I will note the remarkable fact on the energy functional (2.6). For the variation of $\bar{t}$ obeying the CMKdV equation, the Bernoulli–Euler functional (2.6) is invariant

$$\begin{aligned} -\partial_{\bar{t}} \int \mathrm{d}s |\kappa(s,t,\bar{t})|^2 \\ = \int \mathrm{d}s \partial_s \left(\frac{3}{4}|\kappa|^4 + (\partial_s^2\bar{\kappa})\kappa + \bar{\kappa}\partial_s^2\kappa - |\partial_s\kappa|^2 \right) = 0, \end{aligned} \tag{2.33}$$

as the NLS flows conserve the first integral

$$\partial_t \int \mathrm{d}s |\kappa(s,t,\bar{t})|^2 = -\mathrm{i} \int \mathrm{d}s\, \partial_s((\partial_s\bar{\kappa})\kappa - \bar{\kappa}\partial_s\kappa) = 0. \tag{2.34}$$

Now I will consider the measure of the partition function. Since the CMKdV and the NLS problems are initial value problems, for *any* regular shape of elastica satisfied with the boundary conditions, the "virtual time" developments of the curvature of t and $\bar{t}$ are uniquely determined. As described above, the "time" $\mathrm{d}t$ and $\mathrm{d}\bar{t}$ are expected as orthogonal in the solution space of the CMKdV and the NLS equations. It means that for a given regular curve, there exist individual families of the solutions of the CMKdV (2.30) and the NLS (2.31) equations which contain the given curve as an initial condition. Due to relations (2.33) and (2.34), during the motion of t and $\bar{t}$, the Bernoulli–Euler functional (2.6) does not change its value. Hence the deformation parameters t and $\bar{t}$ draw the trajectories of the functional space which have the same value of the Bernoulli–Euler functional (2.6).

In the case that I immersed an elastica in $\mathbb{R}^2$, the thermal fluctuation obeys the MKdV equation and there appears single sort of hierarchy or the MKdV hierarchy [3]. In this article, the codimension of the immersion of the elastica in $\mathbb{R}^3$ is 2 while in the former problem is 1 [3]. Accordingly, it is natural that there appear twice the degrees of freedom of the elastica in $\mathbb{R}^2$, $\{t\}$ and $\{\bar{t}\}$ for the elastica in $\mathbb{R}^3$.

By the "time" development of $\{t\}$ and $\{\bar{t}\}$, I can classify the functional space of the partition function (2.8) by its value E, in which curves are satisfied with the boundary conditions

$$k(0) = k(L), \quad \mathbf{X}_{\mathrm{qcl}}(0) = \mathbf{X}_{\mathrm{qcl}}(L). \tag{2.35}$$

The partition function (2.7) can be represented as

$$\begin{aligned}\mathcal{Z}_{\rm reg} &= \int \mathrm{d}\mu \exp(-\beta E) = \sum_E \exp(-\beta E) \int_{\Xi_E} \mathrm{d}\mu_E \\ &= \sum_E \exp(-\beta E) \mathrm{Vol}(\Xi_E),\end{aligned} \tag{2.36}$$

where $\mathrm{d}\mu = \sum_E \mathrm{d}\mu_E$ and $\mathrm{Vol}(\Xi_E) = \int_{\Xi_E} \mathrm{d}\mu_E$ is the volume of the trajectories Ξ_E of the CMKdV and the NLS hierarchies which occupy the same energy E.

It is known that any solution of the NLS equation (2.31) can be expressed by the hyperelliptic function and its modulus agrees with the modulus of the hyperelliptic curve [6–9,20]. Grinevich and Schmidt [20] studied the moduli of the NLS equation (2.31) whose corresponding space curve is satisfied with the boundary condition (2.35). There appears the finite dimensional Jacobi variety representing the solutions of the closed elastica. As well as the arguments in Ref. [3], even though I introduced the infinite dimensional coordinates $\mathbf{t}$ in (2.22), the infinite dimensional parameters are reduced to finite dimensional one, as the Jacobi variety of a hyperelliptic curve with finite dimension is embedded in the infinite dimensional UGM [8,9]. Here I will evaluate the NLS part of the $(\Xi_E, \mathrm{d}\mu_E)$ as its subspace and submeasure. Using the genus g of the hyperelliptic curves, the NLS part $(\Xi_E^{\rm NLS}, \mathrm{d}\mu_E^{\rm NLS})$ can be decomposed as $(\Xi_E^{\rm NLS}, \mathrm{d}\mu_E^{\rm NLS}) = \coprod_g (\Xi_E^{{\rm NLS}(g)}, \mathrm{d}\mu_E^{{\rm NLS}(g)})$. The NLS part of the measure $\mathrm{d}\mu_E^{{\rm NLS}(g)}$ is locally expressed as $\mathrm{d}t_1 \wedge \mathrm{d}t_3 \wedge \cdots \wedge \mathrm{d}t_{2g-1}$, which is a "real" subspace of (complex) g-dimensional Jacobi variety [6,7]; the restriction of complex space to real one comes from a reality condition [6,20]. As described above, the remaining parameters $\{t_{2n+1} | n \geq g\}$ are not employed in the measure because they are linearly dependent and are not necessary to represent the thermal fluctuation.

For the NLS equation (2.31), there are infinite Jacobi varieties who have the same energy E in general. Thus it is expected that the CMKdV equation is connected among these Jacobi varieties. In other words for each point of the flows obeying the NLS hierarchy (NLS flows), there are (at least parts of) the flows governed by the CMKdV hierarchy (CMKdV flows) which are perpendicular with $\mathrm{d}\mu_E^{{\rm NLS}(g)}$ and conserve the energy. I can locally express the measure of $\mathrm{d}\mu_E^{(g)}$ related to g-dimensional Jacobi varieties, as

$$\mathrm{d}\mu_E^{(g)} = \mathrm{d}\mu_E^{{\rm NLS}(g)} \wedge \mathrm{d}\mu_E^{{\rm CMKdV}(g)}, \tag{2.37}$$

where $\mathrm{d}\mu_E^{{\rm CMKdV}(g)}$ means the measure of the CMKdV part. As the CMKdV equation is expected as a soliton equation, the dimension of the measure $\mathrm{d}\mu_E^{{\rm CMKdV}(g)}$ is also expected as finite dimensional. Thus the integration in the partition function (2.8) is reduced to sum of the ordinary integrations of finite dimension. As curves corresponding to higher genus have larger energy E, the series with respect to the genus in the partition function (2.36) might converge.

Here I will note that the dimension in (2.37) does differ for different genus g. By exchanging the coordinate $\mathrm{d}t_i$ and $\mathrm{d}t_j$ of multi-times t, the volume of $\int \mathrm{d}\mu_E^{(g)}$ is estimated by the unit of the elastica length L. Since the dimension of the Bernoulli–Euler functional E is the inverse of length and $\beta/[\text{length}]$ is order of unity, the multiple of the length can be interpreted as the multiple of the inverse temperature β^{-1}. Hence the sum of terms with different dimensional volume which appear in (2.36) can be regarded as expansion of power of β.

3. Discussion

As well as the two dimensional case in Ref. [3], by investigating the partition function with Bernoulli–Euler functional (2.6), I derived the CMKdV and the NLS hierarchies by physical requirements. It is very surprising that by physically setting up, the soliton hierarchy consistently appears. For example, Doliwa and Santini [14] derived the CMKdV and the NLS hierarchies by some assumptions but their assumption could not be interpreted by physical model of real material. Their approach looks artificial from the point of view of physics even though it exhibits beautiful mathematical aspect. There are many questions, e.g., why the infinite times appear, why t_{2n+1} and t_{2n-1} must be connected with, why $u_c^{(0)}$ in (2.24) must be chosen as $\partial_s k$ or ik and so on; they did such operations by following the axiom which they proposed [14]. However, in this article I found physical answers of these questions as well as the MKdV hierarchy in Ref. [3]. By proposing a physical model and considering its partition function, the soliton theory are reconstructed. For example, in the sense of the statistical mechanics for a system with the infinite degrees of freedom, the infinite times, e.g. $\{t_{2n+1}\}$, are physically interpreted and I naturally introduced them as deformation parameters. As a system of the linear differential equation like the wave equation is described by a vector in infinite dimensional vector space as mode analysis, I showed that a state of the elastica with finite temperature is also represented by a vector in an infinite dimensional vector space like the UGM. By physical requirements and searching for all extremum points in the partition function, I derived such infinite dimensional vector space $\mathbf{t}$ and their algebraic relation. The space might be the UGM for the Sato theory of the soliton equation and, at least, the NLS part is built in the UGM [8,9]. In other words, at least for the part of the NLS hierarchy, physical investigation of a model of observable object reproduces the soliton theory including Sato theory [8,9].

As I gave a calculation scheme of the partition function of elasticas in $\mathbb{R}^3$ in terms of solution space of the CMKdV equation (2.30) and the NLS equation (2.31), I will comment upon the partition function itself from physical viewpoint. Even though I could not give a concrete form of the partition function (2.8), the problem is reduced to the problem of soliton theory. Thus by investigating the CMKdV equation and the NLS equation, the partition function can be evaluated concretely. In fact, Grinevich and Schmidt [20] studied the concrete shape of the curve governed by the NLS equation. Since the Jacobi variety is a finite vector space with algebraic properties and embedded in the UGM, the integral, at least of the NLS part, in the partition function is reduced to sum of the integral over the Jacobi varieties. Further the CMKdV equation is also anticipated as soliton equation and its integration might be finite dimensional one. Since the energy functional is larger for higher genus of the hyperelliptic curve, it is expected that the partition function (2.36) converges as a series with respect to the genus. Hence, even though I introduced the infinite dimensional parameters $\mathbf{t}$, the partition function could be approximated by finite sum by truncating the solution spaces with the genus g, $g \leq G$. Hence in near future, I expect that the value of the partition function will be concretely computed. Since the partition function (2.8) is related to the deoxyribonucleic acid (DNA) problem as a supercoil elastic model [31], I hope that this progress will affect the study of the DNA problem. In other words, I believe that this formulation might shed a new light upon the study of the polymer science [18].

Here I will mention the knot configuration. Since the NLS and the CMKdV equations are initial value problems, the solution space includes *any* configurations of a space curve in $\mathbb{R}^3$. In other words, they also include *any* knot configurations and so I need not pay any attention upon the ambient isotopy [32]. The topological invariance of this system is related to the fundamental group $\pi(S^1)$ as I showed in [27] rather than knot invariance. In fact the trajectories of the NLS and the CMKdV equations classify space curves immersed in $\mathbb{R}^3$ rather than ones embedded in $\mathbb{R}^3$; crossings are allowed and its topology disables us to distinguish such knot invariances or ambient isotopy. Since the knot configuration is physically discriminated by means of long range force such as the electromagnetic force and this theory in this article does not include such force, this notion can be also physically interpreted. If one wishes to consider the knot configuration in this system, it might be related to the gauged NLS equation [33].

As described above, the partition function (2.36) is a map from the configuration space of the immersed curve to real one-dimensional space $\mathbb{R}$. Any curves with different topology are summed up. I showed that classifying the configuration space (functional space $\mathbf{X}(s)$) by energy E is equivalent to investigation of the orbit space of the NLS equation and the CMKdV equation. As they are kinematic system of the virtual times, two points in orbit space with the same energy are transitively acted by a transformation group and for such group action, the partition function is invariant. Thus the partition function (2.36) might be regarded as a character of the group and thus should be studied from group theory [8,9].

Next I will give two comments on the CMKdV equation. First, one might have a question why I need the CMKdV hierarchy whereas the solution space of the NLS hierarchy includes any configurations of a space curve in $\mathbb{R}^3$, at least as an initial condition. I have been dealing with the measure of the functional space. An uncountable set of $\mathbb{R}$ becomes $\mathbb{R}^2$ if the elements are measurable and one can define $\mathbb{R}^2$ topology in the set. In the similar meaning, I need the CMKdV hierarchy in order to introduce the natural measure in the functional space of the two-codimensional immersion object. In other words, they are complementary objects.

The solutions of the NLS hierarchy are described in terms of the hyperelliptic functions [6–9,20]. A hyperelliptic curve is embedded in a Jacobi variety. The trajectories of the NLS hierarchy $(t_1, t_3, \ldots, t_{2g-1})$ have the vector structure as the Jacobi variety. The NLS flow covers a subset of Jacobi variety. The individual Jacobi varieties are distinguished by points in the Siegel upper half space [6–9]. Since the CMKdV flows are expected to have perpendicular part with the NLS flows from the measure theory as described above, the CMKdV flows might connect the different Jacobi varieties of solutions of the NLS equation. Thus I will conjecture, as the second comment upon the CMKdV equation, that the solution space of the CMKdV equation might be realized in the Siegel upper half space related to hyperelliptic function [7].

Finally I will mention the higher dimensional elastica problem, e.g., an elastica in n-dimensional space $\mathcal{C} \subset \mathbb{R}^n$. The codimension of the elastica becomes $n-1$ and thus instead of $\mathbf{t} = (\{t\}, \{\bar{t}\})$, there appear $(n-1)$ sets of infinite dimensional parameters $\mathbf{t} = (\{t^{(1)}\}, \{t^{(2)}\}, \dots, \{t^{(n-1)}\})$. As there appeared $U(1)$-bundle in this article, they represent the $(n-2)$-dimensional inner sphere of sphere bundle over the elastica $\mathcal{C}$ and the normal radius direction of $\mathcal{C}$. Thus there is naturally a principal bundle over $\mathcal{C}$. In other words, one can add the group structure over the equations. Thus the generalized MKdV equation naturally appears [14,34] and it is expected that my computation scheme of the partition function can be extended. Since the elastica in $\mathbb{R}^2$ and $\mathbb{R}^3$ is related to a string (in the string theory) in $\mathbb{R}^3$ [16,20], the elastica in $\mathbb{R}^{25}$ or $\mathbb{R}^{26}$ might be connected with a string in $\mathbb{R}^{24}$. Thus I will expect further progress of this study.

Acknowledgements

I would like to thank Y. Ônishi, Prof. T. Tokihito, Prof. K. Nishinari, Prof. F. Pedit and Prof. S. Saito for critical discussions and continuous encouragements and Prof. B.G. Konopelchenko for helpful comment, sending me his interesting works and encouragement. Especially, Prof. K. Nishinari pointed out that the CMKdV equation has nontrivial N-soliton solutions. I am grateful to Prof. P. Grinevich for sending me his interesting works. I also acknowledge that the seminars on differential geometry, topology, knot theory and group theory with Prof. K. Tamano influenced this work. I also thank M. Nino, T. Hanamura and Prof. A.L. Kholodenko for encouragements.

Appendix A

In this appendix, let us derive the classical shape of the Bernoulli–Euler functional with isometry condition [2]. Since the isometry deformation was discussed in (2.9)–(2.19), deformation along the parameter t is automatically satisfied with the isometry condition. Thus according to the minimal (variational) principle, I will evaluate $\delta E/\delta(\epsilon u_0) = 0$.

By direct computation, (2.19) is expressed by

$$\epsilon(\bar{\kappa} Q^2(u_c) + \kappa \overline{Q^2(u_c)}) = -|\kappa|^2(\bar{\kappa} u_c + \kappa \overline{u_c}) - (\partial_s^2 \bar{\kappa} u_c + \partial_s^2 \kappa \overline{u_c}). \tag{A.1}$$

Here I suppressed the notion of "qcl". Using the relation (2.14), there is an ambiguity between u_0 and u_c, which is related to the boundary condition. I will use ϵu_0 as variational parameter and evaluate the energy functional in terms of u_0,

$$\frac{\delta E}{\delta \epsilon u_0} = \delta_s \left(|\kappa|^2 + \frac{1}{\kappa}\delta_s^2 \kappa \right) = 0. \tag{A.2}$$

Then by integrating (A.2), I obtain the static NLS equation

$$iC_0\partial_s\kappa + \tfrac{1}{2}|\kappa|^2\kappa + \delta_s^2\kappa = 0. \qquad (A.3)$$

This means the governing equation of the classical shape of elastica, which is a special case of the NLS equation (2.31).

References

[1] C. Truesdell, The influence of elasticity on analysis: the classic heritage, Bull. Amer. Math. Soc. 9 (1983) 293–310.
[2] A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, Cambridge, 1927.
[3] S. Matsutani, Statistical mechanics of elastica on plane: origin of MKdV hierarchy, J. Phys. A 31 (1998) 2705–2725.
[4] R.E. Goldstein, D.M. Petrich, The Korteweg–de Vries hierarchy as dynamics of closed curves in the plane, Phys. Rev. Lett. 67 (1991) 3203–3206.
[5] R.E. Goldstein, D.M. Petrich, Solitons, Euler's equation, and vortex patch dynamics, Phys. Rev. Lett. 67 (1992) 555–558.
[6] E.D. Belokols, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, V.B. Matveev, Algebro-Geometric Approach to Nonlinear Integrable Equations, Springer, New York, 1994.
[7] D. Munford, Tata Lectures on Theta, vols. I and II, Birkhäuser, Boston, 1983–1984.
[8] M. Jimbo, T. Miwa, Solitons and infinite Lie algebras, Publ. RIMS, Kyoto Univ. 19 (1983) 943–1001.
[9] M. Sato, Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, in: H. Fujita, P.D. Lax and G. Strang (Ed.), Nonlinear Partial Differential Equations in Applied Science, Kinokuniya/North-Holland, Tokyo, 1983.
[10] M. Lakshmanan, Continuum spin system as an exactly solvable dynamical system, Phys. Lett. A 61 (1977) 53–56.
[11] S. Matsutani, The geometrical construction of the Hirota bilinear form of the modified Kortewegde–Vries equation on a thin elastic rod: bosonic classical theory, Int. J. Mod. Phys. A 10 (1995) 3109–3130.
[12] S. Matsutani, On the relation between modified KdV solitons and Dirac fields on a thin elastic rod, Thesis in Tokyo Metropolitan University, 1996.
[13] K. Nishinari, Nonlinear dynamics of solitary waves in an extensible rod, Proc. Roy. Soc. London A 453 (1997) 817–833.
[14] A Doliwa, P.M. Santini, An elementary geometric characterization of the integrable motions of a curve, Phys. Lett. A 185 (1994) 373–384.
[15] J. Langer, R. Perline, Poisson geometry of the filament equation, J. Nonlinear Sci. 1 (1991) 71–91.
[16] S. Matsutani, On density of state of quantized Willmore surface: A way to a quantized extrinsic string in $\mathbb{R}^3$, J. Phys. A 31 (1998) 3595–3606.
[17] M. Doi, S.F. Edwards, The Theory of Polymer Dynamics, Oxford University Press, Oxford, 1986.
[18] A.L. Kholodenko, Statistical mechanics of semiflexible polymer: yesterday, today and tomorrow, J. Chem. Soc. Faraday Trans. 91 (1995) 2473–2482.
[19] N. Saitô, K. Takahashi, Y. Yunoki, The statistical mechanical theory of stiff chains, J. Phys. Soc. Jpn. 67 (1967) 219–226.
[20] P.G. Grinevich, M.U. Schmidt, Closed curves in $\mathbb{R}^3$: a characterization in terms of curvature and torsion, the Hasimoto map and periodic solutions of the filament equation, dg-ga 9703020.
[21] J.J. Millson, B. Zombro, A Kähler structure on the moduli space of isometric maps of a circle into Euclidean space, Invent. Math. 123 (1996) 35–59.
[22] A.M. Polyakov, Gauge Fields and Strings, Harwood Academic Publishers, London, 1987.
[23] D. Lovelock, H. Rund, Tensors, Differential Forms, and Variational Principles, Wiley, New York, 1975 (republished in Dover, 1989, New York).
[24] H.W. Guggenheimer, Differential Geometry, Dover, New York, 1963.
[25] H. Hashimoto, A soliton on vertex filament, J. Fluid Mech. 51 (1972) 477–485.
[26] R.C.T. da Costa, Quantum mechanics of a constrained particle, Phys. Rev. A 23 (1981) 1982–1987.

[27] S. Matsutani, Anomaly on a submanifold system–new index theorem related to a submanifold system, J. Phys. A 28 (1995) 1399–1412.
[28] S. Matsutani, On time development of a quasi-quantum particle in quartic potential $(x^2 - a^2)^2/2g$, Rev. Math. Phys. 9 (1997) 943–991.
[29] K. Nishinari, Private communication.
[30] A.A. Mohammad, M. Can, Exact solutions of the complex modified Korteweg–de Vries equation, J. Phys. A 28 (1995) 3223–3233.
[31] R.H. Austin, J.P. Brody, E.C. Cox, T. Duke, W. Volkmuth, Stretch Genes, Phys. Today 1058 (1997) 32–38.
[32] L.H. Kauffman, Statistical mechanics and the Jones polynomial, Contemporary Math. 78 (1988) 263–297.
[33] R. Jackiw, S.-Y. Pi, Soliton solutions to the gauged nonlinear Schrödinger equation on the plane, Phys. Rev. Lett. 64 (1990) 2969–2973.
[34] S. Matsutani, On the physical relation between the Dirac equation and the generalized mKdV equation on a thin elastic rod, Phys. Lett. A 189 (1994) 27–31.